Curvature lines on orthogonal surfaces of $\mathbb{R}^3$ and Joachimsthal Theorem

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Abstract

In this paper is studied, as a complement of Joachimsthal theorem, the behavior of curvature lines near a principal cycle common to two orthogonal surfaces.

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1 Introduction

The local behavior of curvature lines near umbilic points was considered by G. Darboux, [3], for analytic surfaces and by C. Gutierrez and J. Sotomayor, [7], for C^r surfaces.
Near principal cycles, the local behavior of curvature lines was first considered in details by C. Gutierrez and J. Sotomayor, [7]. They obtained the derivative of the first return map $\pi : \Sigma \to \Sigma$ associated to the periodic leaf and showed that generically (open and dense set of immersions) the principal cycles are hyperbolic, i.e, $\pi'(0) \neq 1$.
The Joachimsthal theorem says that two surfaces intersecting at a constant angle along a regular curve γ and this curve is a curvature line of one surface then it is a curvature line of the other.
The main goal of this paper is to describe the local behavior near a principal cycle common to two surfaces intersecting orthogonally.

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2 Differential equation of curvature lines

A *principal curvature line* is a regular curve (parametrized by arc length s) $\gamma:(a,b)\to\mathbb{M}\backslash\mathcal{U}$ such that for all $s\in(a,b)$ we have $\gamma'(s)$ is a principal direction.
The normal curvature at p in the direction $w\in T_p\mathbb{M}$ is $k_n(p;w)=II(p;w)/I(p;w)$, where I and II are, respectively, the first and second fundamental forms of $\mathbb{M}$.
Therefore, $w=(du,dv)$ is a principal direction, if and only if, there exists $\lambda\in\mathbb{R}$ such that

$$II(p;w)=\lambda I(p;w),\quad I(p;w)=1.$$

This means that I e II are proportional in the direction w.
As $I(p;w)=Edu^2+2Fdudv+Gdv^2$ and $II(p;w)=edu^2+2fdudv+gdv^2$ we have that $w=(du,dv)$ is a principal direction, if and only if,

$$\frac{\partial(I,II)}{\partial(du,dv)}=0.$$

Or, equivalently by,

$$(Fg-Gf)dv^2+(Eg-Ge)dudv+(Ef-Fe)du^2=0. \tag{1}$$

In the case where $\mathbb{M}$ is parametrized as graph $(x,y,h(x,y))$ we have that

$$\begin{aligned} E=&1+h_x^2, & F&=h_xh_y, & G&=1+h_y^2,\\ e=&\frac{h_{xx}}{\sqrt{EG-F^2}}, & f&=\frac{h_{xy}}{\sqrt{EG-F^2}}, & g&=\frac{h_{yy}}{\sqrt{EG-F^2}}.\end{aligned}$$

When $\mathbb{M}$ is defined implicitly $\mathbb{M}=\{(x,y,z):h(x,y,z)=0\}$ the differential equation of curvature lines is expressed y

$$[dp,\nabla h,d\nabla h]=0,$$

where $dp=(dx,dy,dz)$, $\nabla h=(h_x,h_y,h_z)$, $d\nabla h=(dh_x,dh_y,dh_z)$ and $[.,.,.]$ denotes the mist product of three vectors.

Remark 1. See the books and lecture notes [1], [2], [5], [7], [6], [8], [9], [10], [11] and [12] for more on local and global properties of principal curvature lines on surfaces.

3 General properties of curvature lines

Theorem 1 (Joachimsthal). Let $\mathbb{M}_1\subset\mathbb{R}^3$ and $\mathbb{M}_2\subset\mathbb{R}^3$ two regular and oriented surfaces such that $\mathbb{M}_1\cap\mathbb{M}_2=\gamma$ is a regular curve and $\langle N_1(\gamma(s)),N_2(\gamma(s))\rangle=cte$ along γ, where N_1 and N_2 are unitary normal vector fields to $\mathbb{M}_1$ and $\mathbb{M}_2$. Then γ is a principal curvature line of $\mathbb{M}_1$ if and only if it is a curvature line of $\mathbb{M}_2$.

Proof. Suppose that $\langle N_1(\gamma(s)), N_2(\gamma(s))\rangle = 0$.
Let $T = \gamma'(s)$ and suppose that γ is a principal curvature line, with geodesic curvature $k_{g,1}$, geodesic torsion $\tau_{g,1} = 0$ and principal curvature $k_{m,1}$, for the surface $\mathbb{M}_1$. See [11]. So,

$$\begin{aligned} T' =& k_{g,1}N_1 \wedge T + k_{m,1}N_1 \\ (N_1 \wedge T)' =& -k_{g,1}T + \tau_{g,1}N \\ N_1' =& -k_{m,1}T - \tau_{g,1}N \wedge T \end{aligned} \tag{2}$$

The Darboux frame for γ, as a curve of $\mathbb{M}_2$, is given by:

$$\begin{aligned} T' =& k_{g,2}N_2 \wedge T + k_{n,2}N_2 \\ (N_2 \wedge T)' =& -k_{g,2}T + \tau_{g,2}N_2 \\ N_2' =& -k_{n,2}T - \tau_{g,2}(N_2 \wedge T)' \end{aligned} \tag{3}$$

where $k_{n,2}$ is the normal curvature, $\tau_{g,2}$ is the geodesic torsion and $k_{g,2}$ is the geodesic curvature of γ as a curve of $\mathbb{M}_2$.
Also $N_2 = \pm N_1 \wedge T$, since $\langle N_1, N_2\rangle = 0$. Suppose $N_2 = N_1 \wedge T$. From the equations (2) and (3), and using that $N_1 = T \wedge N_2$, it follows that:

$$\begin{aligned} \tau_{g,2} =& \tau_{g,1} = 0 \\ k_{g,1} =& k_{m,2} \\ k_{g,2} =& k_{m,1}, \end{aligned}$$

where $k_{m,2}$ is a principal curvature of $\mathbb{M}_2$. Therefore γ is a principal curvature line of $\mathbb{M}_2$.
The case $\langle N_1, N_2\rangle = cte \neq 0$ is analogous. □

Proposition 1. A closed, simple and biregular curve $c : \mathbb{R} \to \mathbb{R}^3$, $|c'(s)| = 1$, of length L and torsion τ is a principal curvature line of a surface if, and only if, $\int_0^L \tau(s)ds = 2k\pi, k \in \mathbb{N}$.

Proof. Consider the Frenet frame $\{t, n, b\}$ associated to c.
Let $N = \cos\theta(s)n(s) + \sin\theta(s)b(s)$ be a unitary normal vector to c.
So it follows that,

$$N'(s) = -k(s)\cos\theta(s)t(s) + (\theta'(s) + \tau(s))[-\sin\theta(s)n(s) + \cos\theta(s)b(s)].$$

Therefore, $N'(s) = \lambda t(s)$ if and only if $\theta'(s) + \tau(s) = 0$.
So $\theta(L) - \theta(0) = -\int_0^L \tau(s)ds$ e $N(L) = N(0)$ if and only if $\int_0^L \tau(s)ds = 2k\pi$, $k \in \mathbb{N}$. □

Proposition 2. Let $\gamma : [0, L] \to \mathbb{R}^3$ be a principal cycle of a surface $\mathbb{M}$ such that $\{T, N \wedge T, N\}$ is a positive frame of $\mathbb{R}^3$. Then the expression

$$\begin{aligned} \alpha(s, v) =& \gamma(s) + v(N \wedge T)(s) \\ &+ \left(\frac{1}{2}k_2(s)v^2 + \frac{1}{6}b(s)v^3 + \frac{1}{24}c(s)v^4 + o(v^4)\right)N(s), \quad -\delta < v < \delta \end{aligned} \tag{4}$$

where k_2 is the principal curvature in the direction of $N \wedge T$, defines a local C^∞ chart on the surface $\hat{\mathbb{M}}$ defined in a small tubular neighborhood of γ.

Proof. The map $\alpha(s,v,w) = c(u) + v(N \wedge T)(s) + wN(s)$ is a local diffeomorphism in a neighborhood of the s axis. For each s, the curve $v \to v(N \wedge T)(s) + w(s,v)N(s)$ is the intersection of the surface $\hat{\mathbb{M}}$ with the plane spanned by $\{(N \wedge T)(s), N(s)\}$. Using Hadamard's lemma it follows that

$$w(s,v) = [\frac{1}{2}k_2(s)v^2 + v^2 A(s,v)]N(s)$$

where $A(s,0) = 0$ and k_2 is the (plane) curvature of the curve in the plane spanned by $\{N \wedge T, N\}$, that cuts the surface $\hat{\mathbb{M}}$. This ends the proof. □

According to [11], the Darboux frame $\{T, N \wedge T, N\}$ along γ satisfies the following system of differential equations:

$$\begin{aligned} T' &= k_g N \wedge T + k_1 N \\ (N \wedge T)' &= -k_g T + 0N \\ N' &= -k_1 T - 0(N \wedge T) \end{aligned} \tag{5}$$

where k_1 is the *principal curvature* and k_g is the *geodesic curvature* of the principal cycle γ.

4 Preliminary calculations

Consider the parametrizations α of $\mathbb{M}_1$ and β of $\mathbb{M}_2$ in a neighborhood of γ, such that $\{T, N \wedge T, N\}$ is a positive frame of γ as a curve of $\mathbb{M}_1$ and $\{T, N, T \wedge N\}$ is a positive frame of γ as a curve of $\mathbb{M}_2$.

$$\begin{aligned} \alpha(s,v) =& \gamma(s) + v(N \wedge T)(s) + [\frac{1}{2}k_2(s)v^2 + \frac{1}{6}b(s)v^3 + O(v^3)]N(s) \\ \beta(s,w) =& \gamma(s) + wN(s) + [\frac{1}{2}m_2(s)w^2 + \frac{1}{6}B(s)w^3 + O(w^3)](T \wedge N)(s). \end{aligned} \tag{6}$$

4.1 Immersion α

The coefficients of the first fundamental form of α are given by:

$$\begin{aligned} E_\alpha(s,v) =& 1 - 2k_g v + [k_g^2 - k_1 k_2]v^2 + O(v^3) \\ F_\alpha(s,v) =& O(v^3) \\ G_\alpha(s,v) =& 1 + k_2^2 v^2 + O(v^3) \end{aligned} \tag{7}$$

The unitary normal vector field $\mathcal{N}_\alpha = (\alpha_s \wedge \alpha_v)/|\alpha_s \wedge \alpha_v|$ is given by:

$$\begin{aligned}\mathcal{N}_\alpha(s,v) =&[-\frac{1}{2}k_2'v^2+O(v^3)]T(s)-[k_2v+\frac{1}{2}b(s)v^2+O(v^3)](N\wedge T)(s)\\&+[1-\frac{1}{2}k_2^2v^2+O(v^3)]N(s)\end{aligned} \tag{8}$$

The coefficients of the second fundamental form of α are given by:

$$\begin{aligned}e_\alpha(s,v) =&k_1-(k_1+k_2)k_gv\\&+\frac{1}{2}[k_2''-(k_1+k_2)k_1k_2-k_gb(s)+2k_g^2k_2]v^2+O(v^3)\\f_\alpha(s,v) =&k_2'v+\frac{1}{2}[k_gk_2'+b'(s)]v^2+O(v^3)\\g_\alpha(s,v) =&k_2+b(s)v+\frac{1}{2}(c(s)-k_2^3)v^2+O(v^3)\end{aligned} \tag{9}$$

The functions $L_\alpha=(Fg-Gf)_\alpha$, $M_\alpha=(Eg-Ge)_\alpha$ and $N_\alpha=(Ef-Fe)_\alpha$ are given by:

$$\begin{aligned}L_\alpha(s,v) =&-k_2'v-\frac{1}{2}(k_gk_2'+b'(s))v^2+O(v^3)\\M_\alpha(s,v) =&k_2-k_1+[(k_1-k_2)k_g+b(s)]v\\&+\frac{1}{2}[(-3k_1k_2^2-3k_gb(s)+c(s)-k_2^3-k_2''+k_1^2k_2]v^2+O(v^3)\\N_\alpha(s,v) =&k_2'v+\frac{1}{2}(b'(s)-3k_gk_2')v^2+O(v^3)\end{aligned} \tag{10}$$

The functions $\mathcal{K}_\alpha$ and $\mathcal{H}_\alpha$ are given by:

$$\begin{aligned}\mathcal{K}_\alpha(s,v) =&k_1k_2+[(k_1k_2-k_2^2)k_g(s)+k_1b(s)]v+O(v^2)\\\mathcal{H}_\alpha(s,v) =&\frac{1}{2}(k_2+k_1)+\frac{1}{2}[(k_1-k_2)k_g+b(s)]v+O(v^2)\end{aligned} \tag{11}$$

The principal curvatures $k_{1,\alpha}=\mathcal{H}_\alpha-\sqrt{\mathcal{H}_\alpha^2-\mathcal{K}_\alpha}$ and $k_{2,\alpha}=\mathcal{H}_\alpha+\sqrt{\mathcal{H}_\alpha^2-\mathcal{K}_\alpha}$ are given by:

$$\begin{aligned}k_{1,\alpha}(s,v) =&k_1+(k_1-k_2)k_gv+0(v^2)\\k_{2,\alpha}(s,v) =&k_2+b(s)v+0(v^2)\end{aligned} \tag{12}$$

Remark 2. The following relations holds

$$k_g(s)=-\frac{(k_1)_v}{k_2-k_1},\quad k_g^\perp(s)=-\frac{(k_2)'}{k_2-k_1},\qquad b(s)=(k_2)_v=\frac{\partial k_2}{\partial v} \tag{13}$$

Here $k_g^\perp(s)$ is the geodesic curvature of the other principal curvature line which pass through $\gamma(s)$.

4.2 Immersion β

The coefficients of the first fundamental form of β are given by:

$$\begin{aligned} E_\beta(s,w) =& 1-2k_1w+(k_1^2+k_gm_2)w^2+O(w^3)\\ F_\beta(s,w) =& O(w^3)\\ G_\beta(s,w) =& 1+m_2^2w^2+O(w^3) \end{aligned} \tag{14}$$

The unitary normal vector field $\mathcal{N}_\beta = \beta_s \wedge \beta_w / |\beta_s \wedge \beta_w|$ is given by:

$$\begin{aligned} \mathcal{N}_\beta(s,w) =& [-\frac{1}{2}m_2'w^2+O(w^3)]T(s)-[m_2w+\frac{1}{2}B(s)w^2+O(w^3)](N\wedge T)(s)\\ &+[1-\frac{1}{2}m_2^2w^2+O(w^3)]N(s) \end{aligned} \tag{15}$$

The coefficients of the second fundamental form of β are given by:

$$\begin{aligned} e_\beta(s,w) =& -k_g-k_1[m_2-k_g]w\\ &+\frac{1}{2}[m_2''-k_1B(s)+2k_1^2m_2+k_g^2m_2+k_gm_2^2]w^2+O(w^3)\\ f_\beta(s,w) =& m_2'v+\frac{1}{2}[k_1m_2'+B'(s)]w^2+O(w^3)\\ g_\beta(s,w) =& m_2+B(s)w+\frac{1}{2}(C(s)-m_2^3)w^2+O(w^3) \end{aligned} \tag{16}$$

The functions $L_\beta = (Fg-Gf)_\beta$, $M_\beta = (Eg-Ge)_\beta$ and $N_\beta = (Ef-Fe)_\beta$ are given by:

$$\begin{aligned} L_\beta(s,w) =& -m_2'w-\frac{1}{2}(k_1m_2'+B'(s))w^2+O(w^3)\\ M_\beta(s,w) =& m_2+k_g+[B(s)-k_1(m_2+k_g)]v\\ &+\frac{1}{2}[(3k_gm_2^2-3k_1B(s)+C(s)-m_2^3-m_2''-k_g^2m_2]w^2+O(w^3)\\ N_\beta(s,w) =& m_2'(s)v+\frac{1}{2}(B'(s)-3k_1m_2')w^2+O(w^3) \end{aligned} \tag{17}$$

The functions $\mathcal{K}_\beta$ and $\mathcal{H}_\beta$ are given by:

$$\begin{aligned} \mathcal{K}_\beta(s,w) =& -k_gm_2-[(k_gm_2+m_2^2)k_1+k_gB(s)]w+O(w^2)\\ \mathcal{H}_\beta(s,w) =& \frac{1}{2}(m_2-k_g)+\frac{1}{2}[B(s)-(k_g+m_2)k_1]w+O(w^2) \end{aligned} \tag{18}$$

The principal curvatures $k_{1,\beta} = \mathcal{H}_\beta - \sqrt{\mathcal{H}_\beta^2-\mathcal{K}_\beta}$ and $k_{2,\beta} = \mathcal{H}_\beta + \sqrt{\mathcal{H}_\beta^2-\mathcal{K}_\beta}$ are given by:

$$\begin{aligned} k_{1,\beta}(s,w) =& -k_g-(k_g+m_2)k_1w+O(w^2)\\ k_{2,\beta}(s,w) =& m_2+B(s)w+O(w^2) \end{aligned} \tag{19}$$

5 Principal cycles

Proposition 3 (Gutierrez-Sotomayor). Let γ be a principal cycle of an immersion $\alpha : \mathbb{M} \to \mathbb{R}^3$ of length L. Denote by π_α the first return map associated to γ. Then

$$\begin{aligned}\pi'_\alpha(0) =& exp[\int_\gamma \frac{-dk_2}{k_2-k_1}] = exp[\int_\gamma k_g^\perp(s)ds] \\ =& exp[\int_\gamma \frac{-dk_1}{k_1-k_2}] = exp[\frac{1}{2}\int_\gamma \frac{d\mathcal{H}}{\sqrt{\mathcal{H}^2-\mathcal{K}}}].\end{aligned} \tag{20}$$

Proof. Suppose that γ is a principal cycle and consider the chart (s, v) as defined by the expression of α in the equation (6). The differential equation of the principal curvature lines is given by

$$(f - k_1F)ds + (g - k_1G)dv = 0. \tag{21}$$

Therefore $\pi(v_0) = v(L, v_0)$, where $v(s, v_0)$ is the solution of equation 21 with initial condition $v(0, v_0) = v_0$.
Differentiation of equation 21 with respect to v_0 gives:

$$\frac{d}{ds}(\frac{\partial v}{\partial v_0})(s, v(s, v_0)) = -[\frac{f-k_1F}{g-k_1G}]_v(s, v(s, v_0))\frac{\partial v}{\partial v_0}(s, v(s, v_0))$$

Denote $a(s) = (\frac{\partial v}{\partial v_0})(s, 0)$. Therefore at $v(s, 0) = 0$ it is obtained

$$\frac{d}{ds}a(s) = -\frac{f_v(s,0)}{g-k_1}a(s) = -\frac{k'_2}{k_2-k_1}a(s) = k_g^\perp(s)a(s), \quad a(0) = 1.$$

Integration of the linear differential equation above leads to the result. □

The following result established in [4] is improved in the next proposition.

Proposition 4. Let γ be a principal cycle of length L of a surface $\mathbb{M} \subset \mathbb{R}^3$. Consider a chart (s, v) and a parametrization α as defined by equation (6). Denote by k_1 and k_2 the principal curvatures of $\mathbb{M}$. Suppose that $Jac(k_1, k_2) = \frac{\partial(k_1,k_2)}{\partial(s,v)} = (k_1)_s(k_2)_v - (k_1)_v(k_2)_s \neq 0$ for all $s \in [0, L]$. Then if γ is not hyperbolic then it is semihyperbolic. That is, if the first derivative of the first return map π associated to γ is one, then the second derivative of π is different from zero. In fact, if $\pi'(0) = 1$ then,

$$\pi''(0) = \int_0^L e^{-\int_0^s \frac{k'_2}{k_2-k_1}du}\frac{Jac(k_1,k_2)}{(k_2-k_1)^2}ds.$$

Proof. The differential equation of the principal curvature lines 21 in the chart (s, v) is given by

$$\begin{aligned}\frac{dv}{ds} &= -\frac{f - k_1F}{g - k_1G}\\ &= -\frac{k_2'}{k_2 - k_1}v - \frac{1}{2}[\frac{b'(k_2 - k_1) - 2k_2'b + k_gk_2'(k_1 - k_2)}{(k_2 - k_1)^2}]v^2 + v^2R(s, v). \qquad (22)\\ &= P(s)v + \frac{1}{2}Q(s)v^2 + R(s, v)v^2, \qquad R(s, 0) = 0\end{aligned}$$

Therefore $\pi(v_0) = v(L, v_0)$, where $v(s, v_0)$ is the solution of equation (22) with initial condition $v(0, v_0) = v_0$.
Differentiating twice the equation (22) with respect to v_0 and evaluating at $v_0 = 0$ the following holds

$$\begin{aligned}\frac{d}{ds}(\frac{\partial v}{\partial v_0}) &= P(s)\frac{\partial v}{\partial v_0}\\ \frac{d}{ds}(\frac{\partial^2 v}{\partial v_0^2}) &= P(s)\frac{\partial^2 v}{\partial v_0^2} + Q(s)(\frac{\partial v}{\partial v_0})^2\\ \frac{\partial v}{\partial v_0}(0) &= 1, \qquad \frac{\partial^2 v}{\partial v_0^2}(0) = 0\end{aligned}$$

So,

$$\begin{aligned}\pi''(0) &= \frac{\partial^2 v}{\partial v_0^2}(L) = \int_0^L exp(\int_0^s P(u)du)Q(s)ds\\ &= \int_0^L exp(-\int_0^s \frac{k_2'}{k_2 - k_1}du)[\frac{2k_2'b - b'(k_2 - k_1) - k_gk_2'(k_1 - k_2)}{(k_2 - k_1)^2}]ds\end{aligned}$$

Integration by parts and using that $k_g(k_1 - k_2) = \frac{\partial k_1}{\partial v}$ it follows that

$$\begin{aligned}\pi''(0) &= \int_0^L exp(-\int_0^s \frac{k_2'}{k_2 - k_1}du)[\frac{k_1'\frac{\partial k_2}{\partial v} - k_2'\frac{\partial k_1}{\partial v}}{(k_2 - k_1)^2}]ds\\ &= \int_0^L exp(-\int_0^s \frac{k_2'}{k_2 - k_1}du)\frac{Jac(k_1, k_2)}{(k_2 - k_1)^2}ds\end{aligned}$$

□

Proposition 5. Let $c : \mathbb{R} \to \mathbb{R}^3$, $|c'(s)| = 1$ be a closed, simple and biregular curve of length L and torsion τ such that $\int_0^L \tau(s)ds = 2k\pi, k \in \mathbb{N}$. Then there exists an immersion $\alpha : [0, L] \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ such that $\alpha(s, 0) = c(s)$ is a hyperbolic principal cycle of α.

Proof. It follows from propositions 2 and 3 defining the principal curvatures adequately.
□

Theorem 2. Let γ be a hyperbolic (minimal) principal cycle of a surface $\mathbb{M} \subset \mathbb{R}^3$ of length L. Let k_1 and k_2 the principal curvatures of $\mathbb{M}_1$ and k_g the geodesic curvature of γ. Let $P(s) = k_2'/(k_2 - k_1)$ and suppose that the linear differential equation $f' = P(s)f + k_g'$ has a $L-$periodic solution such that $f(s) \neq 0$ for all $s \in [0, L]$. Then there exists a surface $\mathbb{M}_2 \subset \mathbb{R}^3$ such that γ is a principal hyperbolic principal cycle of $\mathbb{M}_2$ which is orthogonal to $\mathbb{M}_1$ along γ and $\pi_1'(0) = \pi_2'(0)$.

Proof. Consider the parametrizations α of $\mathbb{M}_1$ and β of $\mathbb{M}_2$ in a neighborhood of γ,

$$\begin{aligned}\alpha(s,v) =& \gamma(s) + v(N \wedge T)(s) + [\frac{1}{2}k_2(s)v^2 + \frac{1}{6}b(s)v^3 + O(v^3)]N(s)\\ \beta(s,w) =& \gamma(s) + wN(s) + [\frac{1}{2}m_2(s)w^2 + \frac{1}{6}B(s)w^3 + O(w^3)](T \wedge N)(s).\end{aligned}$$

where $\{T, N \wedge T, N\}$ is a positive frame of γ as curve of $\mathbb{M}_1$ and $\{T, N, T \wedge N\}$ is a positive frame of γ as curve of $\mathbb{M}_2$.
By proposition 3 it follows that

$$\pi_\alpha'(0) = exp[-\int_\gamma \frac{dk_2}{k_2 - k_1}], \qquad \pi_\beta'(0) = exp[-\int_\gamma \frac{dm_2}{m_2 + k_g}] \tag{23}$$

Suppose that the following equation holds

$$\frac{k_2'}{k_2 - k_1} = \frac{m_2'}{m_2 + k_g}.$$

Then m_2 is a defined by the linear differential equation:

$$m_2' - \frac{k_2'}{k_2 - k_1}m_2 - k_g\frac{k_2'}{k_2 - k_1} = 0, \;\; m_2(0) = m_0. \tag{24}$$

The solution of the linear equation above is given by

$$m_2(s) = e^{\int_0^s a(t)dt}[m_0 + \int_0^s e^{-\int_0^t a(u)du}k_g(t)a(t)dt],$$

where $a(s) = k_2'/(k_2 - k_1)(s)$.
As, by hypothesis, $\int_0^L \frac{k_2'}{k_2 - k_1} \neq 0$ it follows that
$m_0 = m_2(0) = m_2(L)$ if and only if

$$m_0 = \frac{\int_0^L (e^{-\int_0^t a(u)du})k_g(t)a(t)dt}{e^{-\int_0^L \frac{k_2'}{k_2 - k_1}ds} - 1}.$$

Therefore the immersion β can be constructed with m_2, principal curvature of β, defined by the equation 24. To finish we need to show that $m_2(s) + k_g(s) \neq 0$ for all $s \in [0, L]$ and so γ is a principal cycle of β.
In the differential equation (24) let $f = k_g + m_2$. So it is obtained,

$$f' = \frac{k_2'}{k_2 - k_1} f + k_g'. \tag{25}$$

By the same argument above the differential equation (25) has a $L-$ periodic solution.
The points s where $f(s) = 0$ correspond to umbilic points of $\mathbb{M}_2$. Therefore γ is a principal cycle of $\mathbb{M}_2$ if equation (25) has a periodic solution which is different from zero for all $s \in [0, L]$. □

Remark 3. The condition $k_g \neq cte$ is a necessary condition for existence of the surface $\mathbb{M}_2$ as stated in the theorem 2 above.

Theorem 3. *Let γ be a minimal principal cycle of a surface $\mathbb{M}_1 \subset \mathbb{R}^3$ such that $k_g|_\gamma \neq cte$. Then there exists a surface $\mathbb{M}_2 \subset \mathbb{R}^3$ such that γ is a principal hyperbolic principal cycle of $\mathbb{M}_2$ which is orthogonal to $\mathbb{M}_1$ along γ.*

Proof. By theorem 1 we have that $-k_g$ is a principal curvature of $\mathbb{M}_2$ having $T \wedge N$ as positive normal vector in a neighborhood of γ. Defining a non constant $L-$periodic function m_2 such that $m_2(s) + k_g(s) > 0$ and $\int_0^L \frac{m_2'}{m_2+k_g} ds \neq 0$ the result follows, observing that $\int_0^L \frac{m_2'}{m_2+k_g} ds = \int_0^L \frac{-k_g'}{m_2+k_g} ds$. □

Theorem 4. *Let γ be a hyperbolic (minimal) principal cycle of a surface $\mathbb{M} \subset \mathbb{R}^3$ of length L. Suppose that the geodesic curvature of γ is not constant. Then there exists a surface $\mathbb{M}_2 \subset \mathbb{R}^3$ such that γ is a hyperbolic principal principal cycle of $\mathbb{M}_2$ which is orthogonal to $\mathbb{M}_1$ along γ.*

Proof. By theorem 1 we have that $-k_g$ is a principal curvature of $\mathbb{M}_2$ having $T \wedge N$ as positive normal vector in a neighborhood of γ. Define a non constant $L-$periodic function m_2 such that $m_2(s)+k_g(s) > 0$ and $\int_0^L \frac{m_2'}{m_2+k_g} ds \neq 0$. Therefore γ is a hyperbolic (minimal) principal cycle of $\mathbb{M}_2$ parametrized in a neighborhood of γ by the parametrization β. Observing that $\int_0^L \frac{m_2'}{m_2+k_g} ds = \int_0^L \frac{-k_g'}{m_2+k_g} ds$, we can define $\bar{m} = m_2 + \epsilon k_g'$ to obtain $\bar{m}$ as a maximal principal curvature of $\mathbb{M}_2$ with $\bar{m} + k_g > 0$ and $\int_0^L \frac{\bar{m}'}{\bar{m}+k_g} ds \neq 0$ for ϵ small.

□

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