A Further Remark on Cauchy's Integral Theorem

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Abstract

An observation on Cauchy's integral theorem is made the purpose of settling a disagreement on the meaning of an inquiry of L. V. Ahlfors.

Key Words: Cauchy's integral theorem, closed and locally exact differential forms, homologous and null-homologous paths.

Resumen

Se hace una observación sobre el teorema integral de Cauchy, destinada a resolver un desacuerdo acerca de una inquietud originada por L. V. Ahlfors.

Palabras Clave: Teorema integral de Cauchy, formas diferenciales cerradas y localmente exactas, curvas homólogas y nul-homólogas.

AMS **Subject Classification:** 30A99.

In an attempt to address an inquiry of L. V. Ahlfors [1], p. 144, we proposed some time ago (see Charris and Rodríguez-Blanco [2]) an analytic proof of Theorema 1 below.

Theorem 1 *If* $w = fdx + gdy$ *is a closed* C^1 *form on an open subset* Ω *of* $\mathbb{C}$ *and if* γ *is a null-homologous closed path in* Ω *then*

$$\int_{\gamma} w = 0 \qquad (1)$$

That w is a *closed* C^1 *form in* Ω means that f, g are C^1 in Ω and

$$\frac{\partial f}{\partial y}(z) = \frac{\partial g}{\partial x}(z), z \in \Omega, \qquad (2)$$

That γ is *null-homologous* in Ω means that if we define

$$\operatorname{Ind}_\gamma(a) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}, \qquad (3)$$

$a \in \mathbb{C} \setminus \alpha([0,1])$, then $\operatorname{Ind}_\gamma(a) = 0$ for all $a \notin \Omega$ (all paths are supposed to be parametrized by [0, 1]).

Now consider the following theorem.

Theorem 2 *If $W = fdx + gdy$ is a locally exact form on the open subset Ω of $\mathbb{C}$, then*

$$\int_\gamma W = 0 \qquad (4)$$

for any closed null-homologous path γ in Ω

That W is a *locally exact form in* Ω means that f and g are continuous in Ω and for any $a \in \Omega$, $r > 0$ and F can be found such that

$D(a,r) = \{z \mid |z - a| < r\} \subseteq \Omega$ and

$$f(z) = \frac{\partial F}{\partial x}(z),\ g(z) = \frac{\partial F}{\partial y}(z),\ z \in D(a,r) \qquad (5)$$

Different persons support the opinion, as they have pointed out to us either personally or by written communication (among them, Professors J. Bustoz, F. Marcellán and F. Soriano, as well as many students), that Ahlfors´ claim refers to THEOREM 2, *not to Theorem 1*.

In order to settle the above disagreement we show in this brief note that *Theorem 2* is a fairly easy consequence of *Theorem 1* and, as a matter of fact, that they are equivalent.

To this purpose, we resort, as we did in [2] for the proof of *Theorem 1*, to the C^∞ functions given by

$$\varphi(z) = \begin{cases} e^{\frac{1}{|z|^2-1}}, & |z| < 1 \\ 0, & |z| \geq 1 \end{cases} \qquad (6)$$

and

$$\varphi\delta(z)=\frac{1}{c\delta^2}\varphi\frac{z}{\delta},\ z\in\mathbb{C},\ \delta>0, \tag{7}$$

where $c=\iint\varphi(z)dxdy,\ z=x+iy.$

Now let f be continuos in Ω and U be an open relatively compact subset of Ω (i. e., $\bar{U}$ is compact and $\bar{U}\subseteq\Omega$). Let

$$f_\delta(\xi):=\iint\limits_{|\xi-z|\le\delta} f(z)\varphi_\delta(\xi-z)dxdy,\xi\in U \tag{8}$$

and $0<\delta<\operatorname{dist}(U,C)$. Observe that Supp $\varphi_\delta=\{z|\,|z|\le\delta\}$. Thus, $f_\delta=f*\varphi_\delta$, the convolution of f and φ_δ on U. Also

$$f_\delta(\xi)=\iint\limits_{|z|\le\delta} f(\xi-z)\varphi_\delta(z)dxdy,\xi\in U, \tag{9}$$

$i.e., f_\delta=\varphi_\delta*f$ on U.

The functions f_δ are C^∞ functions in U with, for example,

$$\frac{\partial f_\delta}{\partial x}(\xi)=\left(f*\frac{\partial\varphi_\delta}{\partial x}\right)(\xi),\ \frac{\partial f_\delta}{\partial y}(\xi)=\left(f*\frac{\partial\varphi_\delta}{\partial y}\right)(\xi),\ldots,etc.$$

Further, $\frac{\partial f_\delta}{\partial x}=\frac{\partial f}{\partial x}*\varphi_\delta$, $\frac{\partial f_\delta}{\partial y}=\frac{\partial f}{\partial y}*\varphi_\delta$, provided $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist in Ω and are continuous there. If K is a compact subset of Ω and $\varepsilon>0$, letting $\delta>0$ be such that $|f(\xi-z)-f(\xi)|\le\varepsilon$ for all $\xi\in K$ and $|z|\le\delta$ (which follows from the uniform continuity of f on any open relatively compact neighborhood U of K in Ω), then

$$|f_\delta(\xi)-f(\xi)|\le\iint\limits_{|z|\le\delta}|f(\xi-z)-f(\xi)|\varphi_\delta(z)dxdy\le\varepsilon\iint\limits_{|z|\le\delta}\varphi_\delta(z)dxdy=\varepsilon,\xi\in K,$$

so that f can be *uniformly approximated on compact subset of* Ω by means of the C^{∞} *functions* f_{δ}. This well known result is of importance in complex analysis and in many other instances. We use it in the proof of *Theorem 2*.

Proof of Theorem 2.

Let $K=\gamma([0,1])\cup I(\gamma)$, where $I(\gamma)=\{a\in\mathbb{C}\setminus\gamma([0,1])|\mathrm{Ind}_{\gamma}(a)=0\}$ is the interior of γ. Since γ is null-homologous in Ω then $I(\gamma)\subseteq\Omega$, and thus K is a compact subset of Ω. Let $\varepsilon>0$ and U be an open neighborhood of K such that $\bar{U}$ is compact and $\bar{U}\subseteq\Omega$. For each $a\in\bar{U}$, let $0<r_a<\mathrm{dist}(\bar{U},\mathbb{C}-\Omega)$ be such that (5) holds for some C^1 function F and all $z\in D(a,r_a)$. Let $r>0$ be a Lebesgue number of the covering $\{D(a,r_a)|a\in\bar{U}\}$, so that, for any $\xi\in\bar{U}$, $D(\xi,r)$ is contained in $D(a,r)$ for some $a\in\bar{U}$, and let $0<\delta<\frac{r}{2}$ be such that $\sup_{z\in K}|f(z)-f_{\delta}(z)|<\varepsilon$ and $\sup_{z\in K}|g(z)-g_{\delta}(z)|<\varepsilon$. We claim that $\mathcal{W}_{\delta}=f_{\delta}dx+g_{\delta}dy$ is a closed form on U. In fact, for $a\in U$ let F be C^1 in $D(a,r)$ and such that (5) holds for all $z\in D(a,r)$. Clearly $(F*\varphi_{\delta})(\xi)$ is well defined for all $\xi\in D(a,\delta)$, and therefore

$$\frac{\partial f_{\delta}}{\partial y}(a)=\left(f*\frac{\partial\varphi_{\delta}}{\partial y}\right)(a)=\left(\frac{\partial F}{\partial x}*\frac{\partial\varphi_{\delta}}{\partial y}\right)(a)=\left(F*\frac{\partial^2\varphi_{\delta}}{\partial x\partial y}\right)(a)$$
$$=\left(F*\frac{\partial^2\varphi_{\delta}}{\partial y\partial x}\right)(a)=\left(\frac{\partial F}{\partial y}*\frac{\partial\varphi_{\delta}}{\partial x}\right)(a)=\frac{\partial g_{\delta}}{\partial x}(a)$$

Hence $\mathcal{W}_{\delta}$ is as claimed. Since $I(\gamma)\subseteq U$, γ is null-homologous in U, and it follows from *Theorem 1* that $\int_{\gamma}\mathcal{W}_{\delta}=0$. so that

$$\left|\int_{\gamma}\mathcal{W}\right|=\left|\int_{\gamma}\mathcal{W}-\int_{\gamma}\mathcal{W}_{\delta}\right|\leq 2\varepsilon L(\gamma),$$

Where $L(\gamma)$ is the lenght of γ. Since $\varepsilon > 0$ is arbitrary, this ensures that (4) holds. □

The argument in the above proof is standard in distribution theory, where the irregular or singular behaviour of relevant functions is transfered to very regular auxiliary ones. In our case, the functions *f*, *g* in *Theorem 2* are not necessarily C^1 and w is not closed, but w_δ is.

Now assume that $w = fdx + gdy$ is a closed C^1 form in Ω and let $a \in \Omega$ and $r > 0$ be such that $D(a,r) \subseteq \Omega$. Let

$$F(z) = \int_{\gamma_z} w, z \in D(a,r), \tag{10}$$

where $\gamma_z(t) = tz + (1-t)z$, $0 \le t \le 1$. A fairly easy calculation shows that (5) holds for $z \in D(a,r)$. Hence w is locally exact in Ω. Thus, *Theorem 2* also implies *Theorem 1*.

Acknowledgments

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References

[1] L. V. Ahlfors, *Complex Analysis*, Third Edition, Mc Graw-Hill, 1979.

[2] J. A. Charris y G. Rodríguez-Blanco, *Sobre el teorema integral de Cauchy*, Boletín de Matemáticas, Nueva Serie, 1 (1994), 1-8.